Int. J. Solids Structures, 1969, Vol. 4, pp. 607 to 627. Pergamon Press. Printed in Great Britain

REGIONS OF EFFECTIVE APPLICATION OF THE METHODS OF THREE-DIMENSIONAL AND TWO-DIMENSIONAL ANALYSIS OF TRANSIENT STRESS WAVES IN SHELLS AND PLATES*

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Abstract—Transient stress waves caused by inputs (applied load, constrained displacements or constrained velocities) acting or increasing to their maximum values during a short time interval are investigated in shells of revolution and in plates within the framework of the linear theory of elasticity. The conclusions on space–time regions of valid and effective application of approximate methods of integration of the equations of the theory of elasticity, approximate equations, and methods of integration of approximate equations are presented.

INTRODUCTION

ACCORDING to the theory of elasticity the boundary regions where the input takes place are considered as sources of initial elementary waves. Reflection of these initial elementary dilatational, shear and head waves from the lateral surfaces of the shell or plate gives rise to a complicated system of reflected elementary waves. The number of reflected elementary waves grows rapidly in the course of time. Therefore, the idea of separate complete description of all these elementary waves practically fails and two groups of approximate methods have been developed. First, methods of investigation of the elementary wave fronts. Second, methods of application of several averaging procedures which do not pay attention to any discontinuities or take into consideration only the main discontinuities. To the second group belong all the methods of application of the two-dimensional shell and plate theories.

Recent surveys of the literature on transient stress waves in shells and plates are given in [1, 2], which contain lengthy reference lists.

In the present paper an extension of the results reported previously for plates in [3–6] and for shells in [7–10] is given. The axially symmetric transient stress waves in shells of revolution having no singular points as well as the cylindrical and plane transient stress waves in plates are considered. For inputs of several types and time-dependence the results of combined application of several methods of both the groups mentioned above are presented with pointing out the space-time regions of valid and effective application of the methods of integration of the equations of the theory of elasticity, approximate equations, and methods of integration of approximate equations.

* Presented at the 12th International Congress of Applied Mechanics, Stanford, California, August 1968.
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EQUATIONS

Let ξ, η be the coordinates on the middle-plane, and let z be the coordinate normal to the middle-plane. Denote by A, B Lamé parameters related to coordinates ξ, η, by R_1, R_2 the radii of curvature of the middle-plane, by t the time, by E the bulk modulus, by ν Poisson's ratio, by $2h$ the thickness of the shell of plate, by c_1, c_2 velocities of the dilatational and shear waves in the theory of elasticity, by c_R the velocity of Rayleigh surface waves, by u_i ($i = 1, 2, 3$) the displacements in ξ, η and z-directions, and by σ^*_{ij} ($i, j = 1, 2, 3$) the stresses.

In what follows, the transient stress waves dependent on t, ξ, z will be considered in problems, in which $u_2 = 0$, $\sigma^*_{22} \neq 0$, $\sigma^*_{12} = \sigma^*_{23} = 0$.

According to the linear theory of elasticity the governing equations are

$$\begin{aligned} &c_1^2 \frac{1}{H_1} \frac{\partial}{\partial \xi}\left\{\frac{1}{H_1 H_2}\left[\frac{\partial}{\partial \xi}(H_2 u_1) + \frac{\partial}{\partial z}(H_1 H_2 u_3)\right]\right\} \\ &\qquad + c_2^2 \frac{1}{H_2} \frac{\partial}{\partial z}\left\{\frac{H_2}{H_1}\left[\frac{\partial}{\partial z}(H_1 u_1) - \frac{\partial}{\partial \xi} u_3\right]\right\} - \frac{\partial^2}{\partial t^2} u_1 = 0, \\ &c_1^2 \frac{\partial}{\partial z}\left\{\frac{1}{H_1 H_2}\left[\frac{\partial}{\partial \xi}(H_2 u_1) + \frac{\partial}{\partial z}(H_1 H_2 u_3)\right]\right\} \\ &\qquad - c_2^2 \frac{1}{H_1 H_2} \frac{\partial}{\partial \xi}\left\{\frac{H_2}{H_1}\left[\frac{\partial}{\partial z}(H_1 u_1) - \frac{\partial}{\partial \xi} u_3\right]\right\} - \frac{\partial^2}{\partial t^2} u_3 = 0, \end{aligned} \tag{1}$$

where

$$H_1 = A(1 + z/R_1), \qquad H_2 = B(1 + z/R_2). \tag{2}$$

Further we shall use the dimensionless quantities

$$\zeta = z/h, \qquad \tau = c_2 t/h, \tag{3}$$

$$u = u_1/h, \qquad w = u_3/h, \qquad \sigma_{ij} = \sigma^*_{ij}(1+\nu)/E, \tag{4}$$

$$k_0 = c_2/c_1, \qquad k_R = c_R/c_2, \qquad k_\pi = [(1-\nu)/2]^{\frac{1}{2}}, \qquad \tau_* = 1/k_\pi \tag{5}$$

We shall also suppose that ξ is a dimensionless coordinate chosen in such a way that $A = h$. Then the equations (1) may be rewritten in the form

$$\begin{aligned} &\frac{\partial^2 u}{\partial \xi^2}\left(\frac{h}{H_1}\right)^2 + \frac{\partial u}{\partial \xi}\frac{h}{H_1}\left[\frac{\partial}{\partial \xi}\left(\frac{h}{H_1}\right) + \frac{1}{H_2}\frac{\partial B}{\partial \xi}\right] + \frac{\partial u}{\partial \zeta} k_0^2\left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2}\right) \\ &\quad + \frac{\partial^2 u}{\partial \zeta^2} k_0^2 - \frac{\partial^2 u}{\partial \tau^2} k_0^2 + \frac{\partial^2 w}{\partial \xi \partial \zeta}\frac{h}{H_1}(1 - k_0^2) + \frac{\partial w}{\partial \xi}\frac{h}{H_1}\left[\frac{1 + k_0^2}{\zeta_1} + \frac{1 - k_0^2}{\zeta_2}\right] \\ &\quad + u\frac{h}{H_1}\left[\frac{\partial}{\partial \xi}\left(\frac{1}{H_2}\frac{\partial B}{\partial \xi}\right) + \frac{h}{R_1}k_0^2\left(\frac{1}{\zeta_2} - \frac{1}{\zeta_1}\right)\right] + w\frac{h}{H_1}\frac{\partial}{\partial \xi}\left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2}\right) = 0, \\ &\frac{\partial^2 w}{\partial \xi^2}\left(\frac{h}{H_1}\right)^2 k_0^2 + \frac{\partial w}{\partial \xi}\frac{h}{H_1}k_0^2\left[\frac{\partial}{\partial \xi}\left(\frac{h}{H_1}\right) + \frac{1}{H_2}\frac{\partial B}{\partial \xi}\right] + \frac{\partial w}{\partial \zeta}\left(\frac{1}{\zeta_1} + \frac{1}{\zeta_2}\right) \\ &\quad + \frac{\partial^2 w}{\partial \zeta^2} - \frac{\partial^2 w}{\partial \tau^2}k_0^2 + \frac{\partial^2 u}{\partial \xi \partial \zeta}\frac{h}{H_1}(1 - k_0^2) + \frac{\partial u}{\partial \zeta}(1 - k_0^2)\frac{1}{H_2}\frac{\partial B}{\partial \xi} \\ &\quad - \frac{\partial u}{\partial \xi}\frac{h}{H_1}\frac{1 + k_0^2}{\zeta_1} - u\left[\frac{1}{H_2}\left(\frac{k_0^2}{\zeta_1} + \frac{1}{\zeta_2}\right)\frac{\partial B}{\partial \xi} + k_0^2\frac{h}{H_1}\frac{\partial}{\partial \xi}\left(\frac{1}{\zeta_1}\right)\right] - w\left(\frac{1}{\zeta_1^2} + \frac{1}{\zeta_2^2}\right) = 0, \end{aligned} \tag{6}$$

where

$$H_1 = h(1+\zeta h/R_1), \qquad H_2 = B(1+\zeta h/R_2), \tag{7}$$

$$\zeta_1 = (1+\zeta h/R_1)R_1/h, \qquad \zeta_2 = (1+\zeta h/R_2)R_2/h. \tag{8}$$

In what follows, we shall use the assumptions

$$B \neq 0, \tag{9}$$

$$\left|\frac{\partial R_j}{\partial \xi}\right| \leq h, \qquad (j = 1, 2), \tag{10}$$

$$|R_j| \geq R_0, \qquad R_0 \gg h, \qquad (j = 1, 2). \tag{11}$$

Then, an approximate discontinuity analysis may be carried out by using the equations

$$\begin{aligned}
&\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial u}{\partial \xi}\frac{1}{B}\frac{\partial B}{\partial \xi}+\frac{\partial^2 u}{\partial \zeta^2}k_0^2-\frac{\partial^2 u}{\partial \tau^2}k_0^2+\frac{\partial^2 w}{\partial \xi \partial \zeta}(1-k_0^2)\\
&\qquad +\frac{\partial u}{\partial \zeta}k_0^2\left(\frac{h}{R_1}+\frac{h}{R_2}\right)+\frac{\partial w}{\partial \xi}\left[\frac{(1+k_0^2)h}{R_1}+\frac{(1-k_0^2)h}{R_2}\right] = 0,\\
&\frac{\partial^2 w}{\partial \xi^2}k_0^2+\frac{\partial w}{\partial \xi}k_0^2\frac{1}{B}\frac{\partial B}{\partial \xi}+\frac{\partial^2 w}{\partial \zeta^2}-\frac{\partial^2 w}{\partial \tau^2}k_0^2+\frac{\partial^2 u}{\partial \xi \partial \zeta}(1-k_0^2)\\
&\qquad +\frac{\partial u}{\partial \xi}(1-k_0^2)\frac{1}{B}\frac{\partial B}{\partial \xi}+\frac{\partial w}{\partial \zeta}\left(\frac{h}{R_1}+\frac{h}{R_2}\right)-\frac{\partial u}{\partial \xi}(1+k_0^2)\frac{h}{R_1} = 0,
\end{aligned} \tag{12}$$

in which only the first and second derivates are taken into consideration and error in the coefficients is of the magnitude h/R_0.

In both equations (12) only the small coefficients of the last two terms are dependent on R_1, R_2. Therefore, the amplitudes of the main discontinuities are almost independent on R_1, R_2.

According to Timoshenko-type theory in problems considered

$$u(\xi, \zeta; \tau) = U(\xi, \tau)+\zeta\psi(\xi, \tau), \qquad w(\xi, \zeta; \tau) = W(\xi, \tau), \tag{13}$$

and the stress–state is described by

$$T_{11} = \int_{-1}^{+1} \sigma_{11}(1+\zeta h/R_2)\,d\zeta, \qquad T_{22} = \int_{-1}^{+1} \sigma_{22}(1+\zeta h/R_1)\,d\zeta, \qquad T_{12} = T_{21} = 0, \tag{14}$$

$$M_{11} = \int_{-1}^{+1} \zeta\sigma_{11}(1+\zeta h/R_2)\,d\zeta, \qquad M_{22} = \int_{-1}^{+1} \zeta\sigma_{22}(1+\zeta h/R_1)\,d\zeta, \qquad M_{12} = M_{21} = 0, \tag{15}$$

$$Q_1 = \int_{-1}^{+1} \sigma_{13}(1+\zeta h/R_2)\,d\zeta, \qquad Q_2 = 0. \tag{16}$$

In this theory we shall denote by k_T the "shear correction coefficient"; in our investigation we take $k_T = k_R$.

As far as the terms with small coefficients h/R_1, h/R_2 are concerned, the equations of Timoshenko-type theory, derived by different authors are different. In problems considered in this paper it was found sufficient to use the following equations of Timoshenko-type theory

$$L_{i1}U + L_{i2}\psi + L_{i3}W = 0, \qquad (i = 1, 2, 3), \tag{17}$$

where

$$L_{11} = P, \qquad L_{12} = L_{21} = Kh/R_1,$$

$$L_{13} = L_{31} = [(1+K)h/R_1 + \nu h/R_2]\frac{\partial}{\partial\xi},$$

$$L_{22} = \tfrac{1}{3}P - K, \qquad L_{23} = -K\frac{\partial}{\partial\xi}, \qquad L_{32} = -K\left(\frac{\partial}{\partial\xi} + \frac{1}{B}\frac{\partial B}{\partial\xi}\right), \tag{18}$$

$$L_{33} = -K\frac{1}{B}\frac{\partial}{\partial\xi}B\frac{\partial}{\partial\xi} + \frac{1-\nu}{2}\frac{\partial^2}{\partial\tau^2} + h^2\left(\frac{1}{R_1^2} + \frac{2\nu}{R_1R_2} + \frac{1}{R_2^2}\right)$$

$$P = \frac{1}{B}\frac{\partial}{\partial\xi}B\frac{\partial}{\partial\xi} + \frac{\nu}{B}\frac{\partial^2 B}{\partial\xi^2} - \left(\frac{1}{B}\frac{\partial B}{\partial\xi}\right)^2 - \frac{1-\nu}{2}\frac{\partial^2}{\partial\tau^2}, \tag{19}$$

$$K = k_T^2\frac{1-\nu}{2}. \tag{20}$$

According to the Timoshenko-type theory, the axially symmetric motions in shells of revolution are described by the sixth order system (17) of the hyperbolic type and three different waves propagate along the ξ-coordinate. Two of these waves propagate with the velocity c_2/k_π, but the third with the velocity c_2k_T, whose numerical value depends on the choice of the "shear correction coefficient" k_T. For plates $R_1 = R_2 = \infty$ and the system (17) splits into a single equation for U

$$PU = 0 \tag{21}$$

and into a system of two equations for ψ, W

$$\begin{aligned} L_{22}\psi + L_{23}W &= 0, \\ L_{32}\psi + L_{33}W &= 0. \end{aligned} \tag{22}$$

Typical special cases are plane waves B = const and cylindrical waves $B = \xi$. In these cases (21) is, in fact, the equation of motion in elementary theory of symmetric deformation of plates.

From Timoshenko-type theory with the aid of well known additional assumptions the Kirchhoff–Love theory and membrane theory may be derived. The system of equations of the Kirchhoff–Love theory of shells is of the parabolic type and the disturbances formally propagate with infinite velocity. According to the membrane theory of shells, in axisymmetric problems there is only one wave that propagates with the velocity c_2/k_π. According to the elementary theory of rods, there is also one wave. However, this wave propagates with a smaller velocity $c_2(2+2\nu)^{\frac{1}{2}}$.

CLASSIFICATION OF STRESS-STATES AND METHODS OF INTEGRATION

Let the dot denote a derivate with respect to τ and the prime denote a derivate with respect to ξ.

We shall assume that the following groups of quantities will be investigated:

$$\begin{aligned} &1.\ u, w, \\ &2.\ u^{\cdot}, u', w^{\cdot}, w', \\ &3.\ \sigma_{11}, \sigma_{22}, \sigma_{13}, \sigma_{33} \end{aligned} \tag{23}$$

We also assume that within these groups of main interest are the quantities having the greatest amplitudes.

Taking into consideration these restrictions we shall introduce the following classification of the stress-states:

Stress-state I: wave-type stress-state, which may be described by the theory of elasticity but not by approximate theories, and in which at least one of the quantities (23) has the amplitudes being of the same order of magnitude as the maximum amplitude of the quantities of its group,

Stress-state II: stress-state, which may be described by Timoshenko-type theory or by less complicated approximate theories,

Stress-states III: stress-state, which may be described by the theory of elasticity but not by approximate theories, and in which no one of the quantities (23) has the amplitudes being of the same order of magnitude as the maximum amplitude of the quantities of its group,

Stress-state IV: quasi-static Saint-Venant boundary effects.

We shall deal mainly with the problems of existence and investigation of the stress-states I, II.

The methods of investigation of the axially symmetric transient stress waves in rotationally symmetric shells as well as axially symmetric and plane stress waves in plates are shown in Fig. 1. Refering to the survey [1], very short comments will be submitted here.

Methods 1–7 may be applied to investigate stress-state I; methods 8–16 we shall consider as methods elaborated for description stress-state II.

The methods 1 and 2 are the main methods of investigation of the elementary waves initiated by sharp inputs. They have been widely used in problems of the theory of elasticity taking their origin from theoretical seismology and are well elaborated for plates in [11–19] (see also the surveys [1, 20]). On the level of the approximate equations (12) these methods may be extended without great additional complication also to axially symmetric problems for cylindrical and spherical shells. However, at $\tau > 1$ the methods 1 and 2 are mainly effective only for discontinuity analysis (method 2, also for the analysis at discrete points under the load). Since the number of elementary wave fronts grows rapidly in the course of time, the space–time regions of effective application of methods 1 and 2 are, of course, limited. At the beginning of the process a rather complete description of displacements, their first derivates and stresses is obtained by the three-dimensional meshes method 5 proposed by the author in [1]. According to this method only the main discontinuities are taken into consideration. In axially symmetric problems for arbitrary rotationally symmetric shells the evolution of these main discontinuities is rather easily established analytically on the basis of equations (12) with an error of the magnitude h/R_0; in some special cases the exact formulae may be obtained. Some examples of application of this method of three-dimensional meshes are published in [4, 7–9]. Another variety of the method of discrete three-dimensional meshes was recently used in bars problems in [21].

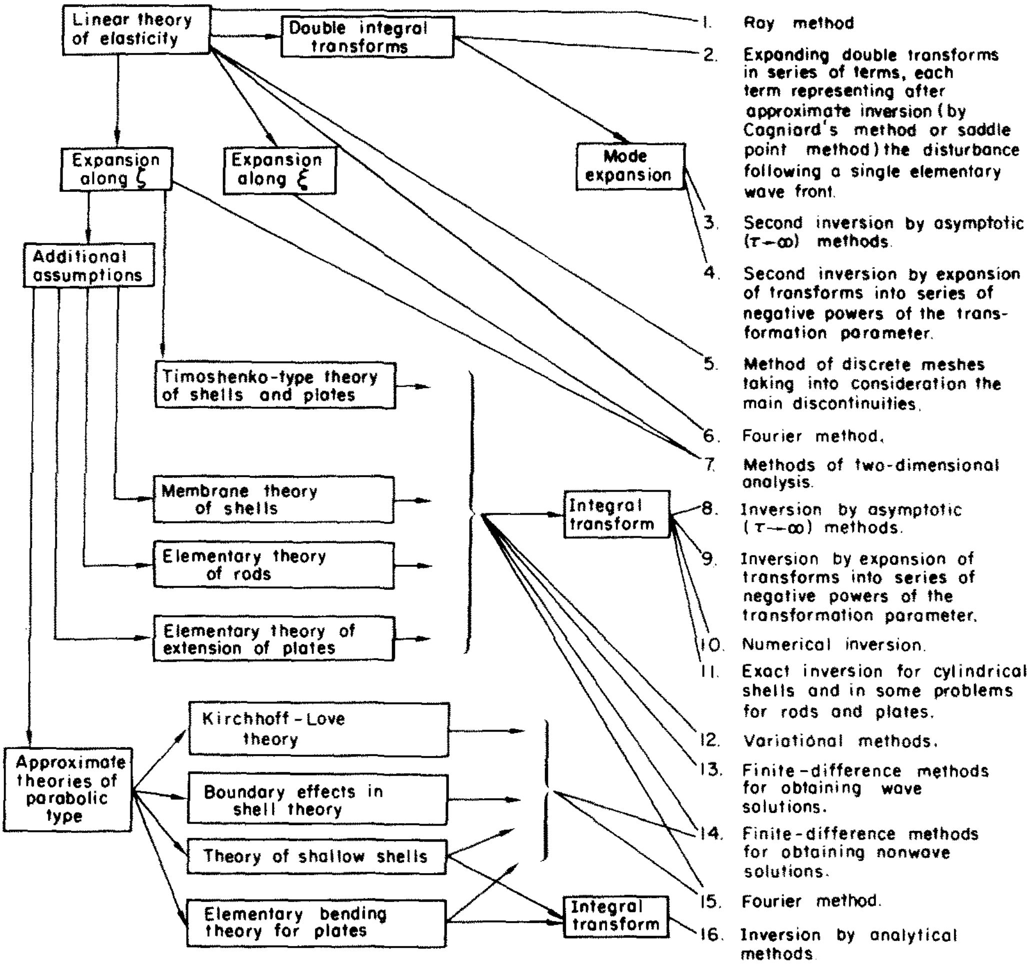

FIG. 1. Methods of integration.

The amount of computation grows by the method 5 in proportion with τ^2. Therefore, at greater values of time it is important to establish the regions of valid application of the other methods shown in Fig. 1. These regions, dependent on input type and time-dependence, will be described in the next section. Here we shall only refer to the literature concerning the methods 3, 4, 6 and 7.

For rods, bars, and plates several varieties of method 3 were elaborated in [3, 22–31]; to cylindrical shells this method was applied in [9, 32]. Method 4 was proposed for plates in [4]. Method 6 was elaborated in [33, 34] for spherical shells in water. Method 7 with expansion along ξ was exploited in [35] in analysis of transient stress waves in spherical and cylindrical shells. Method 7 with expansion along ζ was used for bars in [36]. The references concerning the development of the methods 8–16 are given in surveys [1, 2].

RESULTS

We shall now submit the results of combined application of the methods shown in Fig. 1 in axially symmetric problems for rotationally symmetric shells as well as in axially symmetric and plane problems for plates. The zero initial conditions will be assumed. The transient stress waves initiated by inputs given in the cross section $\xi = 0$ in the form of boundary conditions shown in Table 1 will be considered. Some generalizations will be given at the end of paper. Without restriction of the generality only the stress waves propagating into the positive direction of the ξ-axis will be considered, since the choice of this positive direction is arbitrary. As far as the wave-type solutions obtained by methods 1–13 are concerned, the reflection of the stress waves from the other boundary $\xi = \xi_0$ will not be discussed here.

According to the Table 1 the input time-dependence is defined either through $g_0(\tau)$ or through $g_1(\tau)$. If the displacements are given at $\xi = 0$ through $g_0(\tau)$ then we shall denote

$$g_0'(\tau) = g_1(\tau). \tag{24}$$

Working within the framework of the linear theory of elasticity, it is required that $g_0(\tau)$ is continuous; $g_1(\tau)$ may have finite discontinuities but it is required that $|g_1(\tau)| \ll 1$.

Now we shall define λ through the formula

$$\lambda(\tau) = \frac{|g_1'(\tau)|}{\max|g_1(\tau)|_{0 \leqslant \tau \leqslant \tau_0}} \tag{25}$$

In Table 2 and in explanatory Figs. 2–4 for the four cases of input time-dependence the existence of stress-states I and II, as well as the methods of their investigation are shown. Some comments to this table and to these Figs. will follow.

Case 1

The stress-state I does not exist and of the greatest interest is the stress-state II. The latter is rather slowly varying along ξ and τ; maximum amplitudes appear not earlier than at $\tau \sim \tau_0$, $\tau_0 \gg 1$, when the disturbed region $0 \leq \xi \leq \xi_*$ is large in the sense $\xi_* \gg 1$ or covers the whole middle-plane $0 \leq \xi \leq \xi_0$ if $\tau_0 > k_0\xi_0$. Therefore, in application it is usually quite satisfactory to construct the nonwave solution by methods 14–16, which continue to be effective also at $\tau > k_0\xi_0$ (after reflection of the stress-waves from the boundary $\xi = \xi_0$).

Case 2*

In some sense this case is similar to the Case 1, but the dominating stress-state II varies along ξ and τ more quickly now; maximum amplitudes of the second and third groups of quantities (23) may appear already at rather small values of time $\tau \sim \tau_0$, $\tau_0 \gtrsim 1$. In general, the stress-state II is satisfactorily approximated by wave-type solutions, obtained by methods 8–13 on the basis of the Timoshenko type theory. This theory may be replaced by simpler theories in the same space–time regions as in Case 3.

Case 3

It is the most complicated case and we shall consider the Case 3 separately for inputs *LT*, *LM*, and *NW*. Its limiting case $\tau_0 = 0$ should be understood in the following way:

$$g_1(\tau) = CH(\tau), \qquad C = \text{const.} \tag{26}$$

where $H(\tau)$ denotes Heaviside unit function.

* Here and further $a \lesssim b$ means that a is less but not much less than b.

TABLE 1

Symbol of the input	Boundary conditions in theory of elasticity 1	Boundary conditions in theory of elasticity 2	Boundary conditions in approximate theories 1	Boundary conditions in approximate theories 2	Boundary conditions in approximate theories 3
LT	$u = g_0(\tau)$ or $u^{\cdot}, u'$ or $\sigma_{11} = g_1(\tau)$	$w = 0$ or $\sigma_{13} = 0$	$U = g_0(\tau)$ or $U^{\cdot}, U'$ or $T_{11} = g_1(\tau)$	$\psi = 0$ or $M_{11} = 0$	$W = 0$ or $Q_1 = 0$
LM	$u = \zeta g_0(\tau)$ or $u^{\cdot}, u'$ or $\sigma_{11} = \zeta g_1(\tau)$	$w = 0$ or $\sigma_{13} = 0$	$U = 0$ or $T_{11} = 0$	$\psi = g_0(\tau)$ or $\psi^{\cdot}, \psi'$ or $M_{11} = g_1(\tau)$	$W = 0$ or $Q_1 = 0$
NW	$u = 0$ or $\sigma_{11} = 0$	$w = g_0(\tau)$ or $w^{\cdot} = g_1(\tau)$	$U = 0$ or $T_{11} = 0$	$\psi = 0$ or $M_{11} = 0$	$W = g_0(\tau)$ or $W^{\cdot} = g_1(\tau)$

TABLE 2

Case	Time-dependence of the input	Dominating stress-states and methods of investigation of them
1.	$\lambda \ll 1$ at $0 \le \tau \le \tau_0$ $\tau_0 \gg 1$	Stress-state II; non-wave solutions by methods 14–16.
2.	$\lambda \lesssim 1$ at $0 \le \tau \le \tau_0$ $\lambda = 0$ at $\tau > \tau_0$ $\tau_0 \gtrsim 1$	Stress-state II; wave solutions by methods 8–13.
3.	$\lambda \gg 1$ at $0 \le \tau \le \tau_0$ $\lambda = 0$ at $\tau > \tau_0$ $0 \le \tau_0 \ll 1$	Stress-states I and II in the regions shown for inputs *LT*, *LM*, *NW* in Figs. 2–4; stress-state I by methods 1–5, 7, and stress-state II by methods 8–13.
4.	$\lambda \gg 1$ at $0 \le \tau \le \tau_0$ $g_1(\tau) = 0$ at $\tau > \tau_0$ $\tau_0 \ll 1$	Stress-state I in the same regions as in the case 3; methods 1–3.

Dealing with the Case 3 we shall, for generality, assume that $[B(0)/B(\xi)]^{\frac{1}{2}}$ will not, with growth of ξ, become very small in comparison with unity, since otherwise the stress-state I will vanish and the Case 3 will be at great values of τ similar to the Case 2.

Input LT. Let us consider in the first place axisymmetric transient stress waves in rotationally symmetric shells. In explanations of Fig. 2 the following nomenclature will be used: 1—theory of elasticity; 2—Timoshenko-type theory; 3—membrane theory; 4—boundary effects in the sense of shell theory; 5—stress-state III; 6—Timoshenko-type theory for U only; A, C and F—zones of the stress-state I, E—zone of existence of the stress-state IV.

In the zone F the methods 1 and 2 are effective for revealing elementary wave fronts and discontinuities on them. A first-order approximation of discontinuities may be obtained on the basis of equations (12). According to these equations the main discontinuity moves with velocity c_1 and is carried away by u. At the main discontinuity a first order approximation of u is obtained from the equation

$$u'' + u'B'/B + k_0^2 \partial^2 u/\partial \zeta^2 - u^{\cdot\cdot} = 0, \tag{27}$$

whose solution is almost in proportion with $[B(0)/B(\xi)]^{\frac{1}{2}}$. A rather complete description of the quantities (23) is obtained by method 5. An example of its application is given in Appendix 1. Similarly to the well known results for plates in this and in other examples for shells, next to the main discontinuity by their power and amplitude are the discontinuities on the two or three head-wave fronts behind the main front. On the lateral surfaces these fronts appear at points situated at $\tau/k_0 - \xi \approx j\tau_*/k_0$ ($j = 1, 2, 3$).

It is an interesting problem for further investigations to establish, whether variational method described in [37] is effective in zone F.

Already at $\tau \approx 5 \div 10\tau_*$ the stress-state *I* is localized in narrow zones A and C. Consequently, the other methods are more effective at $\tau > 5 \div 10\tau_*$: in zone A—methods 1 and 4, in zone C—method 3, behind zone C—Timoshenko-type theory (as far as U is concerned this theory may be applied also in region 6). But there exist possibilities for further simplifications in the regions, where Timoshenko-type theory is applicable.

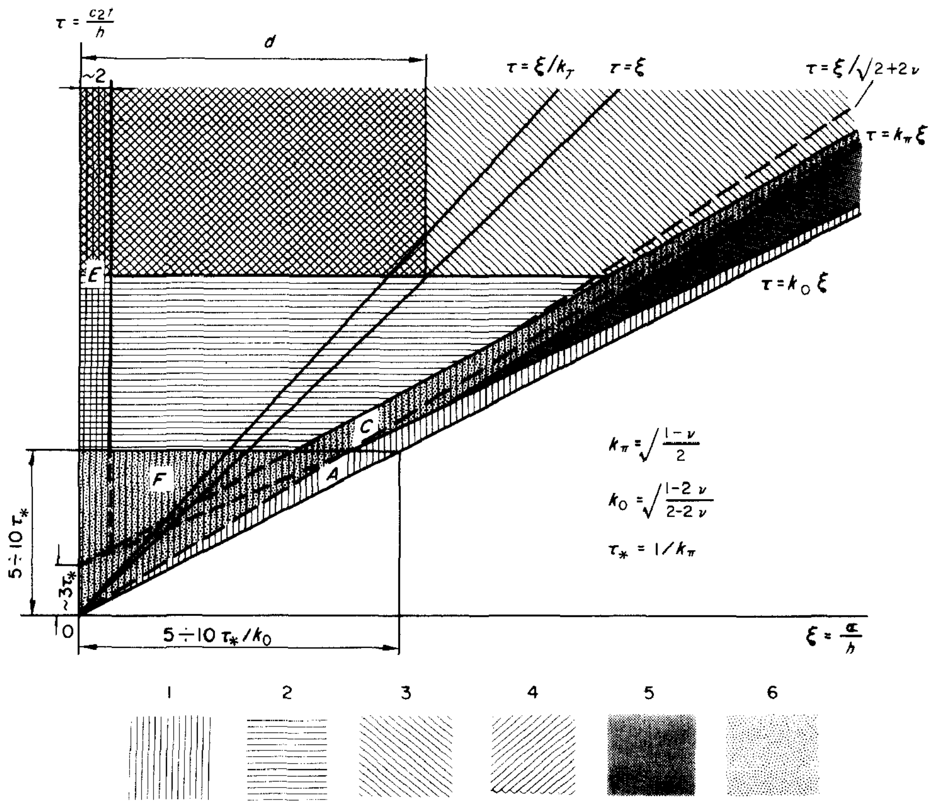

FIG. 2. Input LT with time-dependence according to Case 3: 1—theory of elasticity, 2—Timoshenko-type theory, 3—membrane theory, 4—boundary effects, 5—stress-state III, 6—Timoshenko-type theory for U only; A, C and F—zones of the stress-state I, E—zone of existence of the stress-state IV.

At small τ the contribution of ψ and W to the first equation of system (17) of the Timoshenko-type theory is very small and, therefore, a good approximation of U is obtained from equation (21). If B is a slowly variable function then according to equation (21)

$$U \approx [B(0)/B(\xi)]^{\frac{1}{2}} U_0, \tag{28}$$

where U_0 is the solution of the equation

$$U_0'' - \frac{1-\nu}{2} \ddot{U}_0 = 0 \tag{29}$$

Therefore, in the Timoshenko-type theory at the beginning of the process U is different in several LT-type shell and plate problems only with respect to $[B(0)/B(\xi)]^{\frac{1}{2}}$. Since W is very small in comparison with U, W may also be neglected in the formulae for T_{11} and T_{22}. Some numerical results supporting these conclusions are published in [8] for cylindrical shells and in [38] for toroidal, conical and cylindrical shells.

For thick shells a similar situation also exists at $\tau \gg 1$, but then far behind of its front $\tau = \xi/(2+2\nu)^{\frac{1}{2}}$ better approximation is obtained with the aid of the elementary theory of rods with variable cross-section (see Fig. 1).

For thin shells W grows in time (in comparison with U) rather quickly, and, therefore, the approximate methods described above are not valid at $\tau \gg 1$. However, at $\tau > d$, where d denotes the width of the zone of the boundary effects in shell theory (see Fig. 2), another simplification works well: the solution by Timoshenko-type theory may be replaced by membrane solution and dynamic boundary effects. The latter conclusion was established already in 1961 in the paper [39].

The information submitted above may be used to choose a suitable system of functions for effective application of the method 7.

In plates the input LT initiates only a symmetric (over the middle-plane) state of stress and the situation at $\tau > 5 \div 10\tau_*$ is different behind the zone C, since now the equation (21) is applicable there.

Input LM. In explanations of the Fig. 3 with following nomenclature will be used: 1—theory of elasticity, 2—Timoshenko-type theory, 3—Kirchhoff–Love theory of shells or elementary bending theory for plates, 4—stress-state III, 5—Timoshenko-type theory for W only; A, D and F—zones of the stress-state I, E—zone of existence of the stress-state IV.

As far as the problem of choice of methods is concerned the situation in the zones A and F is quite similar to that in the case of input LT. At $\tau > 5 \div 10\tau_*$ behind the zone A

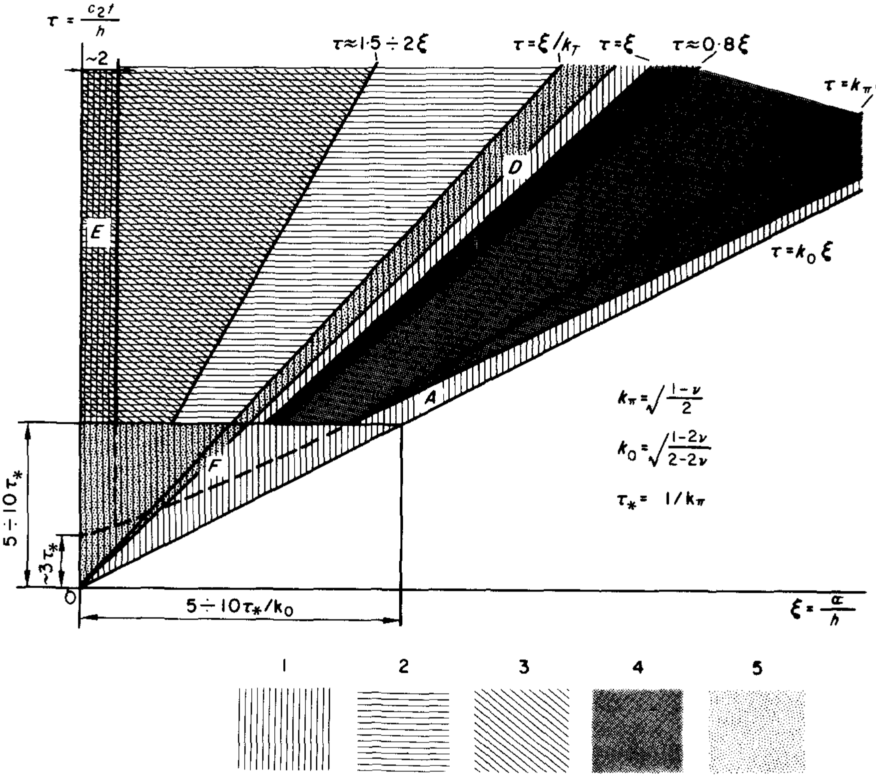

FIG. 3. Input LM with time-dependence according to Case 3: 1—theory of elasticity, 2—Timoshenko-type theory, 3—Kirchhoff–Love theory of shells or elementary bending theory for plates, 4—stress-state III, 5—Timoshenko-type theory for W only; A, D and F—zones of the stress-state I, E—zone of existence of the stress-state IV.

exists a rather large region of small-amplitude oscillations (stress-state III). However, in this region at any $\tau = \text{const}$ far behind the front $\tau = k_0\xi$ the amplitudes begin to grow again. Already at $\tau \sim 0{\cdot}8 \div 0{\cdot}9\xi$ some of the quantities of the second and third groups (23) may have amplitudes being in the same order of magnitude as their maximum amplitudes, that is, zone D of the stress-state I appears, in which method 3 is the main tool of investigation. Except for a rough approximation for W in region 5, Timoshenko-type theory of shells and plates gives satisfactory results in regions 2 and 3. However, in region 3 this theory may be replaced by the Kirchhoff–Love theory in shell problems and by elementary bending theory in plate problems.

At the beginning of motion of shells the contribution of U to the second and third equations (17) is negligible. Consequently, equations (22) may be used not only in plate problems but also in shell problems. According to equations (22) the difference between plates and several shells appears only through B. Therefore, the well-known results for plane ($B = \text{const}$) and cylindrical ($B = \xi$) transient bending waves in plates (see [1, 4, 5]) predict the character of shell motions at the beginning of the process. For example, the cylindrical shell behaves itself like a plate in plane strain and axisymmetric deformation of shallow shells is similar to axisymmetric deformation of plates. With growth of τ the influence of R_1, R_2, U becomes more essential. However, in shells the boundary effects (in the sense of shell-theory) are dominant in region 3. At $\tau \gg 1$ in this region the solution may be constructed as the sum of membrane stress-state and boundary effects.

An example is given in Appendix 2.

Input NW. In explanations Fig. 4 will be used, in which the nomenclature, as far as the numbers from 1 to 5 and zones F and E are concerned, is the same as in Fig. 3. The main difference between the stress-states caused by inputs NW and LM is connected with the fact, that now the main discontinuity propagates with velocity c_2 and is carried away by w. A first-order approximation of discontinuities may be again obtained on the basis of approximate equations (12). However, the evolution of the main discontinuity as well as the evolution of the most essential discontinuities on the elementary wave fronts near to the main front, is roughly described by equation

$$w'' + w'B'/B + \partial^2 w/\partial\zeta^2 - w^{\cdot\cdot} = 0, \tag{30}$$

whose solution is almost in proportion with $[B(0)/B(\xi)]^{\frac{1}{2}}$. Since the main discontinuity moves with the velocity c_2, ahead of zone B, only the stress-state III exists. In zone B mainly methods 1, 3, and 4 should be applied. In zone D method 3 is effective. The regions of effective application of the approximate theories are similar to those in the case of input LM.

Case 4

Displacements u and w are small. Therefore, of main interest are the second and third group of quantities (23). Except in the zone F and Saint-Venant boundary effects in the zone E, most essential amplitudes of these quantities occur in zones A and C if the input LT is applied, in zone A if the input LM is applied, and in zone B if the input NW is applied. Since the essential amplitudes are mainly connected with elementary wave fronts, the methods 1 and 2 are, in general, the most effective tools of investigation. However, at $\tau \gg 1$ the asymptotic method 3 should be applied in the zone C if the input LT is applied. At the beginning of the process methods 5 and 7 are also valid. The approximate theories are not applicable.

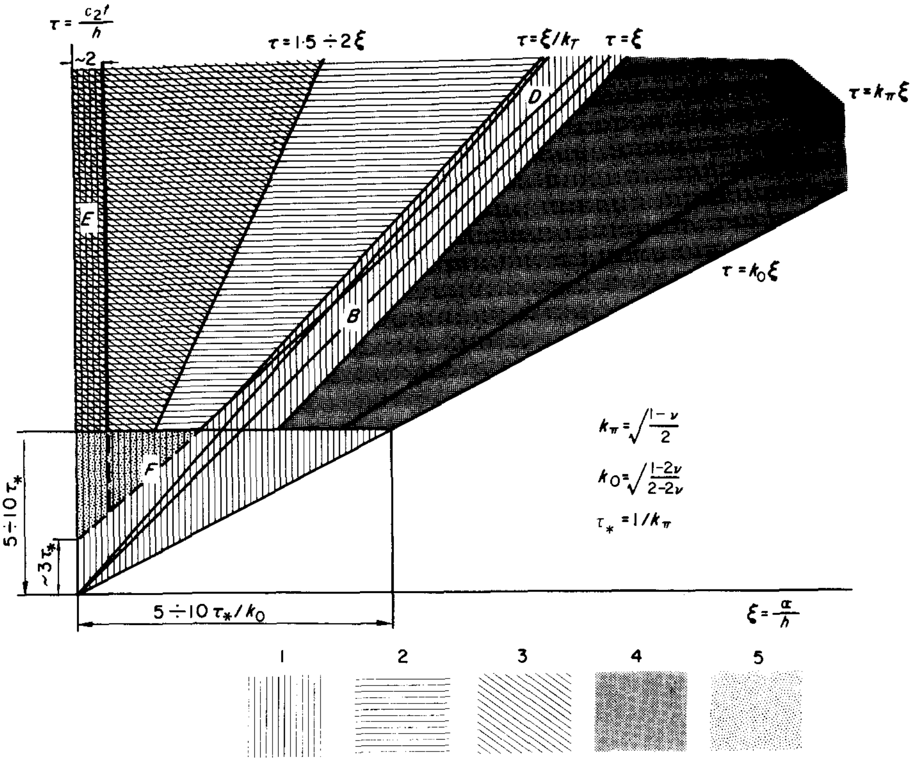

FIG. 4. Input NW with time-dependence according to Case 3: 1—theory of elasticity, 2—Timoshenko-type theory, 3—Kirchhoff–Love theory of shells or elementary bending theory for plates, 4—stress-state III, 5—Timoshenko-type theory for W only; B, D and F—zones of the stress-state I, E—zone of existence of the stress-state IV.

Additional remarks

Method 6 does not occur in Table 2 and in our discussion. This method is not effective in any space–time region if the input is applied in a cross section.

We shall discuss now the choice of methods for obtaining wave-type solutions within Timoshenko-type theory. In early papers (see survey [1]) devoted to plate problems, several direct methods 10 were exploited for inversion of the integral transforms. Since in shell problems the number of singularities is great, the extension of these methods to shell problems is not effective. At the beginning of the process of axially-symmetric motion of shells, methods 13 have been most successfully used. For example, the method of discrete meshes taking into consideration the discontinuities revealed by method 9 was proposed in [1, 4, 5] for plates and afterwards applied effectively in several shell problems [8–10, 38]. The discontinuity analysis carried out in [40] should also be mentioned in connection with this method. The method of characteristics is also quite effective, and was applied in [41] in some problems for cylindrical shells. In plate problems method 12 has been successfully used by several authors (see [1, 36]). By all these methods the amount of computations grows rapidly with the time interval being investigated (almost in proportion with τ^2). Therefore, at $\tau \gg 1$ methods 8 and 9 are more effective (in some examples method 8 was found applicable at $\tau \geq 10 \div 40$). Almost the same holds for other approximate theories.

CONCLUSIONS AND GENERALIZATIONS

1. The results of investigation of transient stress waves caused by inputs *LT*, *LM*, and *NW*, being in the thickness of shell or plate in non-equilibrium and varying slowly over the thickness, are typical for such kind of inputs and may be extended also to other cases. The main conclusion is, that in cases of input time-dependence satisfying the condition

$$\lambda(\tau) \lesssim 1 \tag{31}$$

a satisfactory approximation of the whole process is obtained on the basis of approximate theories, and, in the contrary, in cases $\lambda \gg 1$, as a rule space–time regions appear, where application of the equations of the theory of elasticity is necessary. At $\tau \gg 1$ the latter regions become narrow.

2. If the input is in self-equilibrium in cross section, then in cases (31) Saint-Venant effects are the dominating stress-state, but in cases $\lambda \gg 1$ rather big amplitudes of the first derivates and σ_{ij} may appear in narrow near-front zones, in which the approximate theories do not work.

3. Consider now on the lateral surfaces $0 \leq \xi \leq \xi_0$, $\zeta = \pm 1$ a normal load (σ_{33}) distribution or normal velocity $(w^{\cdot})$ distribution given by function $G(\xi)$. Let us define r by formula

$$r(\xi) = \frac{|G'(\xi)|}{\max|G(\xi)|_{0 \leqslant \xi \leqslant \xi_0}} \tag{32}$$

If now the condition (31) as well as the condition

$$r(\xi) \lesssim 1 \tag{33}$$

are satisfied, then Timoshenko-type theory gives a good approximation for the whole process in axially symmetric shell problems and in bending problems for plates. If the conditions (31), (33) are satisfied in a very strong sense ($\lambda \ll 1, r \ll 1$) and membrane theory works well in the similar static problem, then it gives a good approximation also in dynamic problem.

However, violation of conditions (31) at a discrete value of τ and violation of condition (33) only at a discrete value of ξ lead to appearance of different regions, where Timoshenko-type theory does not give a valid approximation. In the former case these regions appear as zones at the main front (see Figs. 2–4), but in the latter case as a zone of Saint-Venant effect at the point where the condition (33) is not satisfied.

If conditions (31) and (33) are satisfied but the shell is not thin in the sense of our basic condition (11), then the method 6 is a rather effective tool of investigation.

The conclusions given here might be extended also to the problems of analysis of reflected waves from elastic shells in an acoustic medium. For this purpose, the conditions given above must be applied to the initial pressure wave.

In an ideal acoustic medium the distribution and time-dependence of velocities are coupled, since they are determined through a linear second-order wave-equation by the boundary condition at the radiation source. Therefore, the function $g_1(\tau)$ as velocity time-dependence at the radiation source must now be restricted to make the approximate shell-theories applicable for the description of the whole process. Let us define $\lambda(\tau)$ again by (25). Then Timoshenko-type theory gives a good approximation, if $\lambda(\tau) \lesssim 1$, provided the shell satisfies the conditions (9)–(11). In [33, 34] on the basis of equations of the theory of elasticity by Fourier's method 6 the numerical data for reflected waves from spherical shells ($R_1 = R_2 = R_0$) in water were obtained assuming that $g_1(\tau) = \sin \omega\tau$ in the time interval $0 < \tau < \tau_0$ and $\tau_0 \gg 1/\omega$. According to our conclusion a good approximation of these solutions may be obtained by many times smaller amount of computation work when a similar version of Fourier's method 15 is used on the basis of Timoshenko-type theory in problems characterized by $h/R_0 \ll 1$ and $\omega = \lambda \lesssim 1$. The asymptotic error of membrane theory in these problems has two estimations, namely $\nu_1 = h/R_0$ and $\nu_2 = R_0\omega^2/h$, and, consequently, gives almost the same results as Timoshenko-type theory if $\omega^2 \leq h/R_0$.

4. We shall discuss now from a more general point of view the possibility of approximation of a wave-type solution of the equations of the theory of elasticity ("A") by a wave-type solution of approximate equations of shell-theory ("B"), assuming that problems are considered, which might be formulated in the approximate theory being investigated. For example, discussing Timoshenko-type theory, the problems, in which the input is in self-equilibrium in cross section, must be excluded.

Since "A" is a system of elementary waves, the number of which grows rapidly in the course of time and most of whose fronts are not normal to the middle-plane, but "B" is an ensemble of a small number of waves, the distribution of which in the thickness of the shell is prescribed by a certain type of slowly-varying continuous functions and whose fronts are normal to the middle-plane, then during the whole process in all space–time regions a valid approximation of "A" by "B" for displacements and their derivatives up to the nth order is possible under the condition that the nth order derivatives in "A" are continuous and slowly varying through the thickness of the shell. This condition can be replaced by restriction that nth order derivatives in "A" are Lipschitz continuous functions of time τ and coordinates ξ, η along the middle-plane, since then they are also Lipschitz continuous functions of the normal coordinate ζ. Thus, for the purpose pointed out above, it is necessary that the nth order derivatives of displacements with respect τ, ξ, η must be continuous and the $(n+1)$th order derivatives of displacements must be limited in some sense dependent on approximate theory being investigated. In shells having smooth* middle-plane these conditions are satisfied at any finite τ, if the initial waves satisfy them before their first reflections from the lateral surfaces, i.e. nth order derivatives of displacements with respect τ, ξ, η must be continuous and $(n+1)$th order derivatives of displacements must be in some sense limited in the region where the input takes place.

In the main body of the paper we limited ourselves through (23) to the investigation of the case $n = 1$ for axially-symmetric and plane problems. Having limited the $n+1 = 2$ order derivatives with respect to τ and ξ in the input region by conditions (31) and (33) we established on the basis of numerical results that a good approximation of displacements and their $n = 1$ order derivatives is obtained during the whole process by Timoshenko-type theory. The discussion given here leads to the opinion that this conclusion holds also for arbitrary n, if instead of (25) and (32) we define $\lambda(\tau)$ and $r(\xi)$ by formulae

$$\lambda(\tau) = \frac{\left|\dfrac{\partial^{n+1}}{\partial\tau^{n+1}} g_1(\tau)\right|}{\max\left|\dfrac{\partial^n}{\partial\tau^n} g_1(\tau)\right|_{0\leq\tau\leq\infty}}, \qquad r(\xi) = \frac{\left|\dfrac{\partial^{n+1}}{\partial\xi^{n+1}} G(\xi)\right|}{\max\left|\dfrac{\partial^n}{\partial\xi^n} G(\xi)\right|_{0\leq\xi\leq\xi_0}}$$

* See conditions (9)–(11) for axisymmetric and plane problems.

To obtain good approximations for displacements and their derivatives up to the nth order during the time interval $0 \leq \tau \leq \tau_0$ by Timoshenko-type theory in problems, in which the process is dependent on time and on all three space coordinates, $(n+1)$th order derivatives of displacements with respect to τ, ξ, η in the input region must evidently be restricted in a similar sense.

REFERENCES

[1] L. AINOLA and U. NIGUL, Волновые процессы деформации упругих плит и оболочек, *Trans. Acad. Sci. Estonian SSR, Ser. Phys. Maths* **14**, 3–63 (1965).
[2] N. A. ALUMYAE, Переходные процессы деформации упругих оболочек и пластин, *Proc. Sixth all-Union Conf. Theory Shells Plates, Baku—1966, Moscow, Nauka* (1966).
[3] U. NIGUL, Применение трехмерной теории упругости к анализу волнового процесса изгиба полубесконечной плиты при кратковременно действующей краевой нагрузке, *Prikl. Mat. Mekh.* **27**, 1044–1056 (1963).
[4] U. NIGUL, О методах и результатах анализа переходных волновых процессов изгиба упругой плиты, *Trans. Acad. Sci. Estonian SSR, Ser. Phys. Maths* **14**, 345–384 (1965).
[5] N. VEKSLER, A. MYANNIL and U. NIGUL, Применение метода сеток в теории типа Тимошенко для исследования переходных волновых процессов деформации плит конечных размеров, *Prȳkl. Mekh.* **1**, 38–49 (1965).
[6] A. MYANNIL and U. NIGUL, О результатах сопоставления метода сеток и метода перевала при анализе переходного волнового процесса деформации плит, *Prikl. Mat. Mekh.* **30**, No. 2 (1966).
[7] U. NIGUL and M. PETERSON, Алгоритм метода трехмерных сеток для анализа динамических переходных процессов осесимметричной деформации цилиндрической оболочки, *Trans. Acad. Sci. Estonian SSR, Ser. Phys. Maths* **15**, 28–35 (1966).
[8] U. NIGUL, О применимости приближенных теорий при переходных процессах деформации круговых цилиндрических оболочек, *Proc. Sixth all-Union Conf. Theory Shells and Plates, Baku—1966, Moscow, Nauka* (1966).
[9] U. NIGUL, Three-Dimensional shell-theory of axially symmetric transient wave in a semi-infinite cylindrical shell, *Archwm Mech. Stosow.* **19**, 839–856 (1967).
[10] N. VEKSLER and U. NIGUL, К теории волновых процессов при осесимметричной деформации сферической оболочки, Инженерный журнал *Механика твердого тела*, **1**, No. 1 (1966).
[11] Динамические задачи теории упругости, Часть I—1951, Часть II—1952, Часть III—1953, Часть IУ—1954, Часть У—1956, Часть УI—1958; *Ученые записки ЛГУ*, No. 149, 162, 170, 177, 208, 246.
[12] Вопросы динамической теории распространения сейсмических волн, Сборник I—1957, Сборник II—1959, Сборник III—1959, Сборник IУ—1960, Сборник У—1961, Сборник УI—1962, Сборник УII—1964, Изд. ЛГУ.
[13] A. G. MENCHER, Epicentrical displacement caused by elastic wave in an infinite slab, *J. appl. Phys.* **24**, 1240–1245 (1953).
[14] L. KNOPOFF, Surface motions of a thick plate, *J. appl. Phys.* **29**, No. 4 (1958).
[15] L. KNOPOFF and F. GILBERT, First motion methods in theoretical seismology, *J. acoust. Soc. Am.* **31**, 1161–1168 (1959).
[16] K. B. BROBERG, A problem on stress waves in an infinite elastic plate, *Trans. R. Inst. Technol., Stockholm*, No. 139 (1959).
[17] N. DAVIDS, Transient analysis of stress-wave penetration in plates, *J. appl. Mech.* **26**, 651–660 (1959).
[18] P. B. BROBERG, Transient analysis of stress-wave penetration in plates, *J. appl. Mech.* **27**, 366–367 (1960).
[19] R. L. ROSENFELD and J. MIKLOWITZ, Wave fronts in elastic rods and plates, *Proc.* 4th *U.S. natn. Congr. appl. Mech.* **1**, 293–303 (1962).
[20] J. MIKLOWITZ, Recent developments in elastic wave propagation, *Appl. Mech. Rev.* **13**, 865–878 (1960).
[21] L. D. BERTHOLF, Numerical solution for two-dimensional elastic wave propagation in finite bars. *J. appl. Mech.* **34**, 725–734 (1967).
[22] R. M. DAVIES, A critical study of the Hopkinson pressure bar. *Trans. R. phil. Soc., Lond.* **A240**, 375 (1948).
[23] R. SKALAK, Longitudinal impact of a semi-infinite circular elastic bar, *J. appl. Mech.* **24**, 59–64 (1957).
[24] R. FOLK, *et al.*, Elastic strain produced by sudden application of pressure to one end of a cylindrical bar, *J. acoust. Soc. Am.* **30**, 552–563 (1958).
[25] G. P. DEVAULT and C. W. CURTIS, Elastic cylinder with free laternal surface and mixed time-dependent end conditions, *J. acoust. Soc. Am.* **34**, 421–432 (1962).

[26] J. Miklowitz, Transient compressional waves in an elastic plate or elastic layer overlying a rigid half-space, *J. appl. Mech.* **29**, 53–60 (1962).
[27] O. E. Jones and A. T. Ellis, Longitudinal strain pulse propagation in wide rectangular bars—I. Theoretical considerations, *J. appl. Mech.* **30**, 51–60 (1963).
[28] R. P. N. Jones, Transverse impact waves in a bar under conditions of plane-strain elasticity, *Q. Jl Mech. appl. Math.* **17**, 401–419 (1964).
[29] R. A. Scott and J. Miklowitz, Transient compressional waves in the infinite elastic plate with a circular cylindrical cavity, *J. appl. Mech.* **31**, 627–634 (1964).
[30] R. L. Rosenfeld and J. Miklowitz, Elastic wave propagation in rods of arbitrary cross section, *J. appl. Mech.* **32**, 290–294 (1965).
[31] O. E. Jones and F. R. Norwood, Axially symmetric cross-sectional strain and stress distribution in suddenly loaded cylindrical elastic bars, *J. appl. Mech.* **34**, 718–724 (1967).
[32] M. V. Aizenberg and L. J. Slepyan, Резонансные волны в полом цилиндре, погруженном в сжимаемую жидкость, Переходные процессы деформации оболочек и пластин, *Proc. all-union symp., Tartu—*1967, *Acad. Sci. Estonian SSR Tallinn,* 13–22 (1967).
[33] R. Hickling, Analysis of echoes from a hollow metallic sphere in water, *J. acoust. Soc. Am.* **34**, 1582–1591 (1964).
[34] K. J. Diercks and R. Hickling, Echoes from hollow aluminium spheres in water, *J. acoust. Soc. Am.* **41**, 380–393 (1967).
[35] H. A. Rahmatulin, *et al.*, Исследование динамики многослойных сферических и цилиндрических упругих оболочек непосредственным интегрированием динамических уравнений теории упругости, Переходные процессы деформации оболочек и пластин, *Proc. all-union Symp., Tartu—*1967, *Acad. Sci. Estonian SSR, Tallinn,* 113–133 (1967).
[36] V. V. Novožilov and L. J. Slepyan, О принципе Сен-Венама в динамике стержней, *Prikl. Mat. Mekh.*, **29**, 261–281 (1965).
[37] L. Ainola, Вариационный принцип динамики линейной теории упругости, *Доклады АН СССР*, **172**, 306–308 (1967).
[38] N. Veksler, Осесимметричные нестационарные процессы деформации оболочек вращения, *Trans. Acad. Sci. Estonian SSR, Ser. Phys. Math.* **17**, 34–40 (1968).
[39] N. A. Alumyae, О применимости метода расчленения напряженного состояния при решении осесимметричных задач динамики замкнутой цилиндрической оболочки, *Trans. Acad. Sci. Estonian SSR, Ser. Phys. Maths,* **12,** 171–181 (1961).
[40] N. Veksler, Исследование фронтовых разрывов при осесимметричной деформации оболочек вращения и круглой плиты, Переходные процессы деформации оболочек и пластин, *Proc. all-Union Symp., Tartu—*1967, *Acad. Sci. Estonian SSR, Tallinn,* 41–49 (1967).
[41] W. R. Spillers, Wave propagation in a thin cylindrical shell, *J. appl. Mech.* **32,** 346–350 (1965).

APPENDIX 1

Numerical results, illustrating axially symmetric stress waves in cylindrical shells ($R_1 = \infty$, $R_2 = R_0 = \text{const}$, $\nu = 0{\cdot}3$, $k_0 = \sqrt{(\frac{2}{7})}$) caused by an end loading at $\xi = 0$ are presented for

Example A:

$$\sigma_{11}(0, \zeta; \tau) = -\frac{1}{1-\nu} H(\tau), \qquad w(0, \zeta; \tau) = 0,$$

where H denotes Heaviside unit function, and for
Example B:

$$\sigma_{11}(0, \zeta; \tau) = -\frac{1}{1-\nu}[H(\tau) - H(\tau - 8/k_0)], \qquad w(0, \zeta; \tau) = 0.$$

In Fig. 5, in two instants the longitudinal cross section of the disturbed part of the shell with the grid used in computations by method 5 is shown. In example A in hatched region of Fig. 5

$$u = \frac{1-2\nu}{1-\nu^2} \frac{\tau - k_0\xi}{k_0} H(\tau - k_0\xi), \qquad w = 0.$$

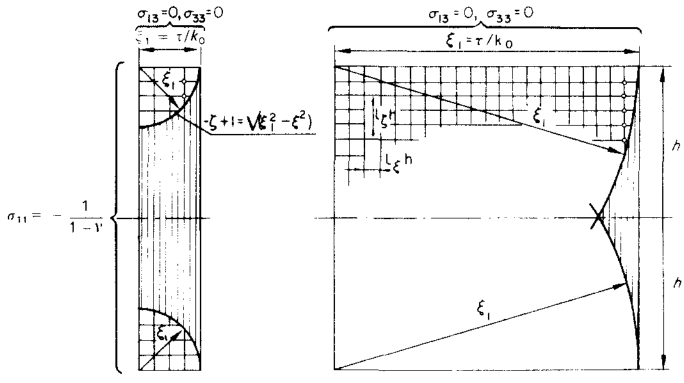

FIG. 5. Example A: longitudinal cross section with grid used in computations by method 5.

Figures 6 and 7 illustrate the results obtained for $\sigma_{11}(\xi, 0; \tau)$ in Examples A and B with $R_0/h = 10$ on the basis of the theory of elasticity by discrete meshes method 5 with a grid $l_\xi = l_\zeta = \frac{1}{10}$, $l_\tau = k_0/20$ (thick line) and on the basis of Timoshenko-type theory by discrete meshes method 13 with a grid $l_\xi = \frac{1}{10}$, $l_\tau = k_0/20$ (thin line).

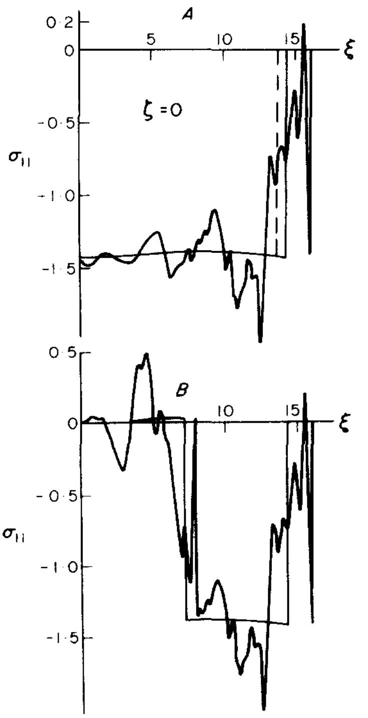

FIG. 6. Examples A and B: thick line—theory of elasticity, thin line—Timoshenko-type theory.

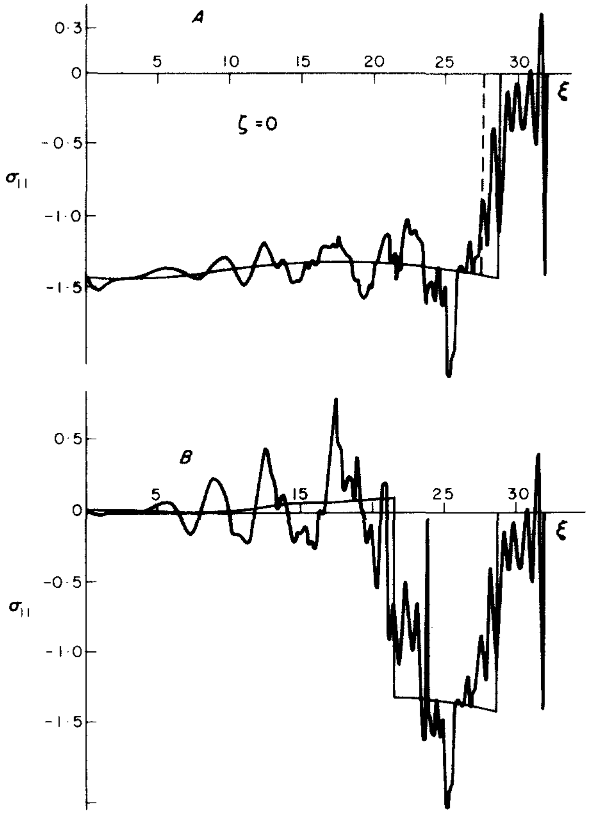

FIG. 7. Examples A and B: thick line—theory of elasticity, thin line—Timoshenko-type theory.

In Fig. 8 the results obtained by several approximate theories are compared in Example C:

$$T_{11} = -H(\tau), \qquad M_{11} = 0, \qquad W = 0$$

At small values of τ max $W \sim$ max Uh/R_0. But note, that already at $\tau = 50\sqrt{\frac{2}{7}}$ max W for a shell $R_0/h = 100$ is only about three times less than max W for a shell $R_0/h = 10$.

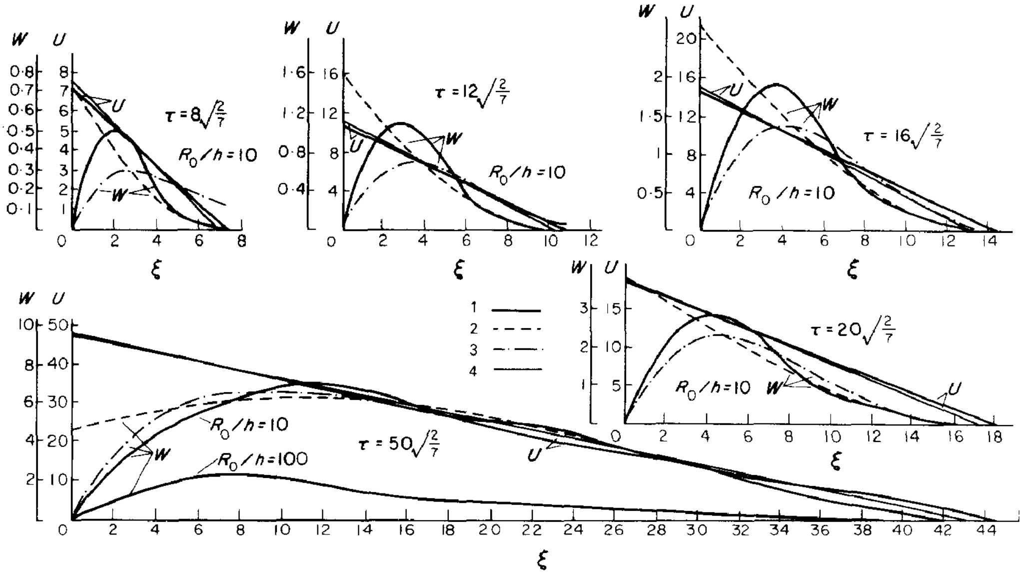

FIG. 8. Example C: 1—Timoshenko-type theory, 2—membrane theory, 3—membrane theory + quasi-static boundary effects, 4—elementary theory of rods.

APPENDIX 2

Numerical results, illustrating plane bending waves in plate ($R_1 = \infty$, $R_2 = \infty$, $\nu = 0{\cdot}3$) caused by an end loading in

Example D:

$$\sigma_{11}(0, \zeta; \tau) = \frac{\zeta}{1-\nu} H(\tau), \qquad w(0, \zeta; \tau) = 0$$

are presented.

In Figs. 9 and 10 the results obtained on the basis of the theory of elasticity by discrete meshes method 5 with a grid (see Fig. 5) $l_\xi = l_\zeta = \frac{1}{19}$, $l_\tau = k_0/38$; on the basis of Timoshenko-type theory by discrete meshes method 13 with a grid $l_\xi = \frac{1}{10}$, $l_\tau = k_0/20$; and by exact formulae of the elementary bending theory are presented. In Fig. 11 the results obtained by saddle point method 8 and discrete meshes method 13 are compared.

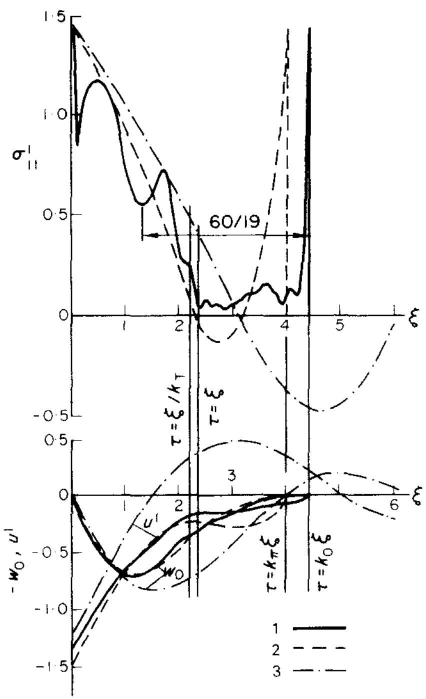

FIG. 9. Example D: 1—theory of elasticity, 2—Timoshenko-type theory, 3—elementary bending theory: $\sigma_{11}^1 = \sigma_{11}(\xi, 1, \tau)$, $u^1 = u(\xi, 1:\tau)$, $w_0 = w(\xi, 0:\tau)$.

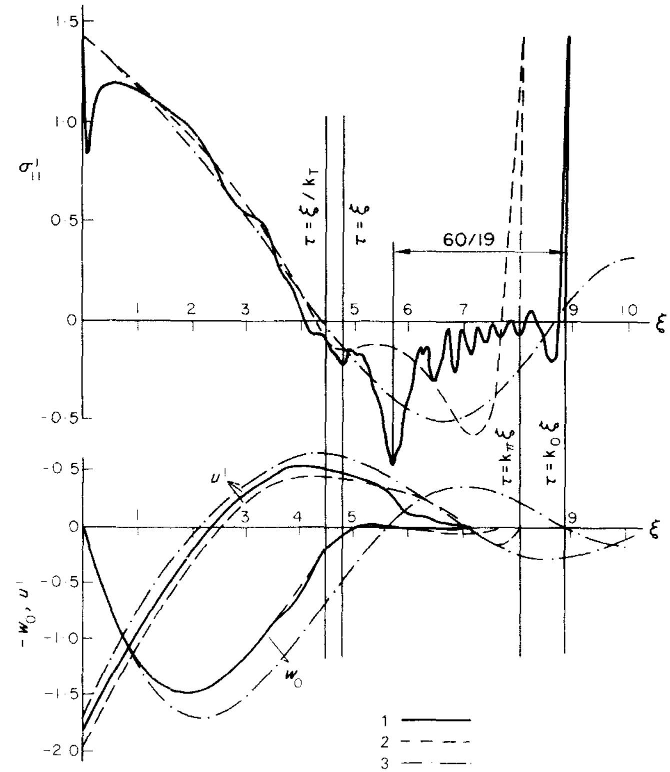

FIG. 10. Example D: 1—theory of elasticity, 2—Timoshenko-type theory, 3—elementary bending theory: $\sigma_{11}^1 = \sigma_{11}(\xi, 1:\tau)$, $u^1 = u(\xi, 1:\tau)$, $w_0 = w(\xi, 0:\tau)$.

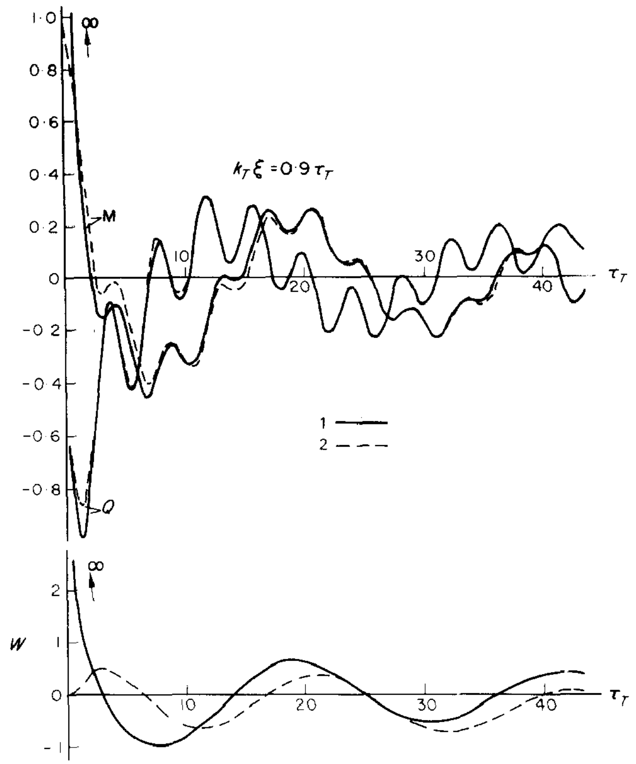

FIG. 11. Example D: results on the basis of Timoshenko-type theory; 1—by saddle point method 8, 2—by discrete meshes method 13.

(*Received* 9 *September* 1968)

Абстракт—В рамках линейной теории упругости исследуются неустановившиеся волны напряжения в оболочках вращения и пластинках, вызванные введением приложенной нагрузки, вынужденных перемещений или вынужденных скоростей. Это введение действует или повышается до своего максимального значения в коротком интервале времени. Представляются результаты, касающиеся пределов значения пространства и времени, эффективного использования приближенных методов интегрирования уравнений теории упругости, а далее приближенные уравнения и методы их интегрирования.